INTEREST CALCULATION AND DIMENSIONAL ANALYSIS

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Abstract. *We treat interest rate as the growth rate of a monetary quantity with dimension* $1/\Delta t$. *With this definition, we can analyze time in discrete and in continuous time interest calculation in a proper manner, and the two interest rate concepts are then consistent too. We study the differences in discrete and continuous time interest and discount calculation, as well as in using different time units in discrete time analysis. We define measurable concepts corresponding to the instantaneous velocities of debt and deposit capitals, and demonstrate their usefulness in economic analysis. (JEL E40, G10).*

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1. Introduction

An overwhelming majority of scientific writings dealing with interest rate treat it as a dimensionless number[2]. It is still assumed in these studies that interest rate is measured per annum, which implies that it is related to time, see e.g. Chiang (1974 p. 289-294) or Hull (2000 p. 51). Treating the interest rate as a pure number makes the comparing of interest rates defined for two days, three weeks, etc. ambiguous, because as pure numbers they ought to be comparable as such. However, we know that 10 % in a week differs from 10 % in a year. Another problem this definition creates is in which units changes in interest rates are measured. If the annual interest rate changes from 10 % to 12 %, the change is 2 percent points or 20 %.

In economic analysis, interest calculation has been defined in discrete and in continuous time. What seems not to have done accurately is the

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[2] Exceptions exist also; de Jong (1967) treats the discrete time interest rate as in here.

treating of time in the analysis. Interest calculation in continuous time is old enough for Samuelson (1937) to call it „*a well-known formula*". Samuelson treats the continuous time interest rate as „*a flow of dollars per unit time divided by the value of the investment*" which definition has not been changed since. Solow (1963) writes: "*The rate of return, as I have defined it, is the rate of interest paid by the bank on one year deposits... ...the rate of return on investment, a dimensionless number (per unit of time)...*". Here we generalize the above definitions so that the time interval the interest rate is measured can be of any length, that is, it does not have to be one year or unity. This generalization – suggested by de Jong (1967) – allows us to deal with interest rate as a dimensional concept.

We study differences in interest and discount calculation in discrete and in continuous time, and in discrete time with varying time units. We search for transformation rules between various interest rate concepts that give identical results by different calculation methods. Such rules have already presented for compound interest calculation, but we show that for discounted money flows and for time paths of debt and deposit capitals, such rules do not exist. Although continuous time is widely used in economic analysis, so far measurable concepts corresponding to the instantaneous velocities of debt and deposit capitals have not been defined. Samuelson (ibid.) though uses these concepts, but does not define them as dimensional quantities. We show that these are important measurable quantities for the continuous time analysis of debt and deposit capitals.

The study is organized as follows. In section 2 we study ways to measure changes in quantities with time. Sections 3 and 4 contain discrete and continuous time interest calculation. In section 5 are given parities between discrete and continuous time interest rate, and between discrete time interest rates of various time units. Section 6 studies the present values of money flows in discrete and in continuous time. Section 7 contains economic analysis of the dynamics of debt and deposit capitals, and section 8 gives a summary.

2. Measuring Changes in Quantities in Time

All monetary quantities belong in the dimension of *monetary values* and are thus additive (de Jong 1967 p. 8). Monetary quantities may not be directly additive, however, because adding 5 (\$) + 10 (£) does not make sense[3]. This adding can be executed if we have a transformation rule for

[3] Measurement units are in parentheses after the numerical values of the quantities.

these two measurement units of value; e.g. 1 (£) = 2.5 ($). Exchange rate $E = 1/2.5\,(£/\$)$ or $1/E = 2.5\,(\$/£)$ expresses this rule. Knowing the exchange rate we can add any two quantities $x(\$)$ and $y(£)$ as $x + y/E$ in units of $ or $xE + y$ in units of £. With a fixed (floating) exchange rate regime, E is a dimensionless constant (varying quantity). These transformations are analogous to those made in adding quantities belonging in the dimension *length* with units *mile* and *meter* by using the rule 1 (mi) = 1609.38 (m).

We assume that time is divided in intervals of equal length Δt that can be measured in any time units: hours (h), days (d), weeks (w), years (y) etc. The transformation rules between the measurement units of the dimension *time* allow us to express Δt in the time units we like. The only problem is that the number of days in one month varies. This can be solved by defining 1 (m) = 30 (d), which gives 1 (y) =12 (m) × 30 (d/m) = 360 (d), or defining 1 (y) = 52 (w), which gives 1 (y) = 52 (w) × 7 (d/w) = 364 (d).

In the following we denote time moments by $t_0, t_1, t_2, \ldots$ and their distances by $\Delta t = t_1 - t_0 \Leftrightarrow t_1 = t_0 + \Delta t,\ t_2 = t_1 + \Delta t = t_0 + 2\Delta t$ etc. Time intervals are named by their ending moments. In discrete time, the length of Δt does not matter; essential is that the quantities applied are measured only at time moments $t_0, t_1, ...$, (or at periods $t_1, t_2, ...$) and not within these moments. This requirement holds for all economic time series. Continuous time is constructed from discrete time by letting $\Delta t \to 0$. This way obtained time units of zero length correspond to time moments, and continuous time is constructed by connecting adjacent time moments.

Interest rate measures the change of a monetary quantity with time. Now, we have four possible ways to measure the change in quantity x during time unit $\Delta t = t_1 - t_0$, where t_0 is a fixed moment:

Table 1.

Discrete time quantities measuring change

$$\begin{aligned}
&\textit{absolute change}: && x(t_0 + \Delta t) - x(t_0),\\
&\textit{relative change}: && \frac{x(t_0 + \Delta t) - x(t_0)}{x(t_0)},\\
&\textit{average velocity}: && \frac{x(t_0 + \Delta t) - x(t_0)}{\Delta t},\\
&\textit{growth rate}: && \frac{x(t_0 + \Delta t) - x(t_0)/\Delta t}{x(t_0)}.
\end{aligned}$$

Table 1 shows that absolute change is measured in the units of x, relative change is dimensionless, average velocity is measured in the units of $x/\Delta t$, and growth rate in the units of $1/\Delta t$. Now, if $\Delta t = 1\ unit\ of\ time$, the numerical values of the first and third, and the second and fourth quantity are equal even though their measurement units differ. The common assumption $\Delta t = 1\ unit\ of\ time$ is the reason that many time interest rate has been erroneously identified as the relative change, and not the growth rate of a monetary quantity, which is the correct definition (see sections 3, 4). In economic data, growth rates are often expressed as growth rates in percent, which are simply growth rates multiplied by 100. Knowing the measurement units of economic quantities is a necessity for well-defined mathematical formulations with these quantities, as we will see later.

The continuous time correspondents for the above discrete time quantities are obtained as their limits with $\Delta t \to 0$. They are in Table 2.

Table 2.
Continuous time quantities measuring change.

$$instantaneous\ absolute\ change: \quad \lim_{\Delta t \to 0} x(t_0 + \Delta t) - x(t_0) = dx \quad at \quad t = t_0\,,$$

$$instantaneous\ relative\ change: \quad \lim_{\Delta t \to 0} \frac{x(t_0+\Delta t)-x(t_0)}{x(t_0)} = \frac{dx}{x} \quad at \quad t = t_0,$$

$$instantaneous\ velocity: \quad \lim_{\Delta t \to 0} \frac{x(t_0+\Delta t)-x(t_0)}{\Delta t} = \frac{dx}{dt} \quad at \quad t = t_0,$$

$$instantaneous\ growth\ rate: \quad \lim_{\Delta t \to 0} \frac{x(t_0+\Delta t)-x(t_0)/\Delta t}{x(t_0)} = \frac{dx/dt}{x} \quad at \quad t = t_0.$$

In mathematics, dx is the differential and dx/dt the time derivative of x. However, pure mathematics differs from “real sciences” there that mathematics does not operate with dimensional quantities. Theoretical physicists, on the other hand, have developed an algebraic theory for mathematical operations with dimensional quantities called *dimensional analysis*. The core of the theory is given e.g. in de Jong (1967). According to dimensional analysis, the measurement units of the above continuous time quantities are identical with their discrete correspondents because taking the limit does not affect the measurement unit.

Let us now assume that quantity x measures the amount of money on a bank account on which the bank pays interest. Because the velocity of the deposited capital depends on the amount of the capital, it is not a good measure for the strength of change of the capital. This problem can be avoided by using measures like the relative change or the growth rate of the capital. For example, with 4 % annual interest rate the annual interest earnings for capitals 100($) and 1000($) are 4($/$y$) and 40($/y), respectively, but the relative changes and growth rates of the two capitals are pairwise equal: 0.04 and 0.04($1/y$). The growth rates of any quantities measured from time interval of length y all have unit $1/y$, and are thus comparable. The growth rate of GDP, the growth rate of the average price level (inflation) e.g. are comparable quantities with interest rate.

3. Interest Calculation in Discrete Time

Suppose that at time moment t_0 amount $x(t_0)(\$)$ is deposited on a bank account with fixed discrete time interest rate i. Interest rate i is the following growth rate of the deposited capital with dimension $1/\Delta t$,

$$\frac{(x(t_0 + \Delta t) - x(t_0))/\Delta t}{x(t_0)} = i \iff x(t_0 + \Delta t) = (1 + i\Delta t)x(t_0).$$

With compound interest calculation, the capital on the bank account after n time units is

$$x(t_0 + n\Delta t) = (1 + i\Delta t)^n x(t_0) \iff x(t_0) = (1 + i\Delta t)^{-n} x(t_0 + n\Delta t)$$

where $(1 + i\Delta t)^n$ and $(1 + i\Delta t)^{-n}$ are the interest and the discount factor between time moments t_0 and $t_0 + n\Delta t = t_n$. The interest and discount factors are dimensionless quantities because the units of i and Δt cancel each other when multiplied together. If the interest rates of the time intervals differ, that is $i_j \neq i_k$, where j refers to the j^{th} time interval, the interest and discount factors become

$$(1 + i_1\Delta t) \times \ldots \times (1 + i_n\Delta t) \ and \ \left((1 + i_1\Delta t) \times \ldots \times (1 + i_n\Delta t)\right)^{-1}.$$

Monetary quantities in all time units belong in the dimension of *value* and are thus additive. However, positive interest rate makes the value of a future U.S. dollar smaller than that of current dollar. In discrete time, the interest and discount factors given above solve this comparability problem between monetary quantities at different time units, similarly as exchange rate solves the comparability problem between two currencies.

4. Interest Calculation in Continuous Time

The continuous time interest rate r is defined analogously with in discrete time as the instantaneous growth rate of a monetary quantity at time moment t, $r(t) = x'(t)/x(t)$. The solution of this differential equation is

$$x'(t) = r(t)x(t) \quad \Leftrightarrow \quad x(t) = x(t_0)e^{\int_{t_0}^{t} r(s)ds} \tag{1}$$

where by s is denoted running time during the interval (t_0, t), and e is the base of the natural logarithm. Eq. (1) expresses the money on a bank account at time moment t when $x(t_0)$ was deposited at moment t_0 with interest rate r that may vary during (t_0, t). In continuous time, we add interest earnings in the capital after every instant of time, which creates the exponential growth. If, however, r is constant, we get

$$\int_{t_0}^{t} rdt = r \times (t - t_0),$$

and the numerical value of $\int_{t_0}^{t} rdt$ equals with that of $r(1/\Delta t) \times \Delta t$ when $t - t_0 = \Delta t$. The present value of $x(t)$ at moment t_0 is obtained from Eq. (1) as

$$x(t_0) = x(t)e^{-\int_{t_0}^{t} r(s)ds}. \tag{2}$$

If $r(s)$ is constant during the interval (t_0, t), Eqs. (1) and (2) become

$$x(t) = x(t_0)e^{r\times(t-t_0)} \quad \Leftrightarrow \quad x(t_0) = x(t)e^{-r\times(t-t_0)}.$$

The continuous time interest and discount factors, $e^{\int_{t_0}^{t} r(s)ds}$ and $e^{-\int_{t_0}^{t} r(s)ds}$, are dimensionless quantities as they ought to be. One difference between the two ways of interest calculation exists, however. In continuous time the length of the time interval $t - t_0$ is measured in time units. Exponent n in discrete time, however, is a pure number that represents the order of the time interval. If time is measured in years in discrete time analysis, then $n = 3$ implies that the interest factor corresponds to the third year from the present moment etc.

5. Parities Between Interest Rates

5.1. Discrete and Continuous Time Interest Rate

According to the previous sections, with compound interest calculation the discrete and continuous time calculated capitals at time moment t_n, deposited on a bank account at moment t_0, are

$$x(t_n) = x(t_0)(1 + i\Delta t)^n \quad and \quad x(t_n) = x(t_0)e^{\int_{t_0}^{t_n} r(t)dt}.$$

Solving $x(t_n)/x(t_0)$ from both equations and setting them equal, yields

$$(1 + i\Delta t)^n = e^{\int_{t_0}^{t_n} r(t)dt}. \tag{3}$$

Thus both methods give identical results if the two interest (and discount) factors are equal. Suppose then that the two interest rates with dimension $1/\Delta t$ are constant. The amount of money on the bank account after time interval Δt, calculated by both methods, is then

$$x(t_1) = x(t_0)(1 + i\Delta t) \; and \; x(t_1) = x(t_0)e^{r\times(t_1-t_0)} \tag{4}$$

Setting these equal and dividing by $x(t_0)$, yields

$$\ln(1 + i\Delta t) = r \times (t_1 - t_0) \quad or \quad i\Delta t = e^{r\times(t_1-t_0)} - 1\,. \tag{5}$$

Using the transformation in Eq. (5), the continuous and discrete time compound interest calculation give identical results (notice that $t_1 - t_0 = \Delta t$ and $r\Delta t$ is a dimensionless quantity with the numerical value of r). The following definition for the continuous time interest rate, conformal with the corresponding discrete time interest rate, can thus be made for time unit $t_1 - t_0$,

$$r_c = \left(\frac{1}{t_1 - t_0}\right)\ln\left(\frac{x(t_1)}{x(t_0)}\right) = \frac{\ln(1 + i\Delta t)}{t_1 - t_0}. \tag{6}$$

Notice that in Eq. (6), $x(t_1)$ is calculated by using the corresponding discrete time interest rate and the discrete time method of calculation.

Example 1. We calculate the capital on a bank account by discrete and continuous time methods when 100($) is deposited at time moment $t_0, t_n - t_0 = 20(y)$ and $i = r = 0.1(1/y)$. The results are graphed in Figure 1. The continuous time capital (the curve) somewhat overestimates the discrete time capital (the dots). However, using $r_c = \ln(1 + i\Delta t)/(t_1 - t_0) = 0.0953(1/y)$ in continuous time, the two time paths coincide, see Figure 2. Hull (2000 p. 52) shows that even without the adjustment $r = r_c$, the continuous time interest calculation gives almost identical results as the daily discrete time analysis with “normal” levels of interest rates.

5.2. Discrete Time Interest Rates of Varying Length

Another parity can be made between discrete time interest rates defined for varying time units. Let us denote the discrete time interest rate for time unit h by i_h, and that for time unit p by i_p, where $s = h/p$ is a positive natural number. Interest returns are assumed to be added in the deposited capital at the ending moment of every time interval. Because $s > 1$ implies $p < h$, then i_p represents a more dense splitting of time. The deposited capital $x(t_0)$ ($) increases in both these regimes during time unit h as

$$x(t_h) = x(t_0)(1 + i_h \times (1h)) \quad and \quad x(t_{sp}) = x(t_0)\left(1 + i_p \times (1p)\right)^s.$$

Setting these two capitals equal and dividing by $x(t_0)$, yields

$$i_{c,h} \times (1h) = (1 + i_p \times (1p))^s - 1 \iff i_{c,p} \times (1p) = (1 + i_h \times (1h))^{\frac{1}{s}} - 1. \quad (7)$$

These transformation rules make any two discrete time interest rates comparable with each other because relation $h = s \times p$ can be defined for every two time units h, p. Eq. (7) thus defines two discrete time interest rates conformal with each other in compound interest calculation, as the subscripts imply.

Example 2. Suppose $i_y = 0.1(1/y), i_m = 0.01(1/m)$ and $1(y) = 12(m)$. Using Eq. (7) the corresponding conformal monthly and annual interest rates are, respectively: $i_{c,m} = 0.00797(1/m)$ and $i_{c,y} = 0.1268(1/y)$. These somewhat differ from the ones we get by using time units in the transformation, $\hat{\imath}_m = 1/12\,(y/m) \times 0.1(1/y) = 0.0083(1/m)$ and $\hat{\imath}_y = 12(m/y) \times 0.01(1/m) = 0.12(1/y)$.

Example 3. Figure 3 shows the differences in the capitals during 20 years when 100 ($) is deposited at time moment t_0 and interest rates $i_y = 0.1(1/y)$ and $\hat{\imath}_m = 0.0083(1/m)$ are applied (the curve refers to monthly and the dots to annual analysis). These differences are shown to disappear in Figure 4 by using the corresponding conformal monthly interest rate, $i_{c,m} = 0.00797(1/m)$.

6. Present Values of Money Flows

Suppose money flow $N(t_0 + j\Delta t)(\$/\Delta t)$, $j = 1,2,\ldots$ and fixed interest rate $i(1/\Delta t)$. In discrete time, the present value of an n period flow is

$$H_n = \frac{N(t_0 + \Delta t)\Delta t}{1 + i\Delta t} + \frac{N(t_0 + 2\Delta t)\Delta t}{(1 + i\Delta t)^2} + \cdots + \frac{N(t_0 + n\Delta t)\Delta t}{(1 + i\Delta t)^n}$$
$$= \sum\nolimits_{j=1}^{n} \frac{N(t_0 + j\Delta t)\Delta t}{(1 + i\Delta t)^j}, \tag{8}$$

where the measurement unit of $N(t_0 + j\Delta t)\Delta t$ is \$ $\forall j$.

Next we analyze the present value of the above flow in continuous time. Discrete time is transformed to continuous by dividing Δt into k equal subintervals and letting $k \to \infty$. At time interval $t_0 + j\Delta t/k$, interest rate $i(t_0 + j\Delta t/k)$ has dimension $1/(\Delta t/k)$ and the corresponding money flow is $N(t_0 + j\Delta t/k)$ with dimension $\$/(\Delta t/k)$, $j = 1,2,\ldots,nk$. With fixed k, the present value of the above flow during nk periods is

$$H_{nk} = \sum\nolimits_{j=1}^{nk} \frac{N(t_0 + j(\Delta t/k))(\Delta t/k)}{[1 + i(t_0 + j(\Delta t/k))(\Delta t/k)]^j}.$$

The discount factor in the formula H_{nk} can be modified as

$$\left[1 + i\left(t_0 + j\frac{\Delta t}{k}\right)\frac{\Delta t}{k}\right]^{-j}$$
$$= \left\{\left[1 + \frac{1}{\frac{k/\Delta t}{i(t_0 + j\,(\Delta t/k))}}\right]^{\frac{k/\Delta t}{i(t_0+j(\Delta t/k))}}\right\}^{\frac{-ji(t_0+j(\Delta t/k))}{k/\Delta t}}$$

see Chiang (1974 p. 290). Now we know that $\lim_{z\to\infty}\left(1 + \frac{1}{z}\right)^z = e$, and because $\lim_{k\to\infty} k/(i(t_0 + j\,\Delta t/k)\Delta t) = \infty$, we can simplify the above formula by the definition of number e. The limiting process transforms the exponent of e to $-r(t) \times (t - t_0)$. This occurs because with $k \to \infty$,

$j\,\Delta t/k \to jdt$ and $i(t_0 + j\,\Delta t/k)j\,\Delta t/k \to r(t_0 + jdt)jdt$. Definition $t = t_0 + jdt, j = 1,2,\dots$ for continuous time completes the proof.

The limiting process $k \to \infty$ transforms the sum in Eq. (8) to an integral with integration limits t_0 and $t_0 + nk\,\Delta t/k = t_0 + n\Delta t = t_n$, and $N(t_0 + j\,\Delta t/k)$ approaches the instantaneous flow $N(t)$ at time moment t. We can thus write

$$\lim_{k\to\infty} H_{nk} = \int_{t_0}^{t_n} N(t)e^{-r(t)\times(t-t_0)dt}, \quad t_0 \le t \le t_n.$$

In order to compare the discrete and continuous time present values, we assume that the flow is fixed, that is, $N(t_0 + j\Delta t) = N, \forall j$. We can then take the factor $N\Delta t$ in front of the sum in Eq. (8) and study the obtained geometric series with positive terms $a^j s$,

$$a^j = \left(\frac{1}{1 + i\Delta t}\right)^j, j = 1,2,\dots,n,$$

where $0 < a < 1$ because $i > 0$. The sum of the series with n terms is

$$S_n = \sum_{j=1}^{n} a^j = \frac{a(1 - a^n)}{1 - a} = \frac{1 - (1 + i\Delta t)^{-n}}{i\Delta t}.$$

The present value of the flow in Eq. (8) with fixed N is then

$$H_n = \frac{N\Delta t[1 - (1 + i\Delta t)^{-n}]}{i\Delta t} = \frac{N}{i}[1 - (1 + i\Delta t)^{-n}].$$

If r is constant, in continuous time the present value of the fixed flow is

$$H(t_n) = \int_{t_0}^{t_n} Ne^{-r\times(t-t_0)}dt = \frac{N}{r}\left[1 - e^{-r\times(t_n-t_0)}\right],$$

where r, t_0, t_n are bounded positive quantities. Because $t_n - t_0 = n\Delta t$, letting $n \to \infty$ we get $H_n \to N/i$ and $H(t_n) \to N/r$. Thus both methods give identical present values for the same infinite fixed flow if $i = r$. Notice that the measurement units of the present values are: $(\$/\Delta t)/(1/\Delta t) = \$$.

The condition for equal present values of the corresponding finite flows is

$$\frac{1}{i}[1 - (1 + i\Delta t)^{-n}] = \frac{1}{r}[1 - e^{-rn\Delta t}]. \qquad (9)$$

An analytic solution for Eq. (9) with respect to i or r does not exist, however, but the equation can be solved numerically keeping n and the other interest rate fixed. These solutions essentially depend on n. In Figure 5 is presented the solutions for $\hat{r}$ with $i = 0.1(1/y)$ when n goes from 5 to 40 with 5 years interval. An increase in n makes $\hat{r}$ closer to $i = 0.1(1/y)$, but the convergent is not fast. We can thus conclude that a unique conformal continuous time interest rate, that gives identical present values as a certain discrete time interest rate, does not exist. With known values of n and i, however, the corresponding $\hat{r}$ can be found numerically.

Assuming $n \to \infty$, we get an asymptotic solution for Eq. (9): $r = i$. Further, the transformation rule for conformal interest rates in compound interest calculation, $r\Delta t = \ln(1 + i\Delta t)$, does not improve the accuracy in calculating present values, and the simple rule $r = i$ gives more accurate results. The reason for this is that the studied money flow is constant and it is not increasing as in compound interest calculation. However, this asymptotic accuracy does not help in calculating present values of money flows for short time intervals.

Next, we calculate the present value of a fixed money flow with varying time units in discrete time. Let the two money flows be $N(\$/h) = N(\$/sp) = \frac{N}{s}(\$/p) = M(\$/p)$, where $h = sp$, and the corresponding interest rates are i_h and i_p, respectively. The present values of the flows are:

$$H_n = \sum_{j=1}^{n} \frac{N \times (1h)}{(1 + i_h \times (1h))^j} = \frac{N}{i_h}[1 + (1 + i_h \times (1h))^{-n}],$$

$$H_{ns} = \sum_{j=1}^{ns} \frac{M \times (1p)}{\left(1 + i_p \times (1p)\right)^j} = \frac{M}{i_p}\left[1 + \left(1 + i_p \times (1p)\right)^{-ns}\right].$$

Setting $H_n = H_{ns}$ and letting $n \to \infty$ we get the asymptotic solution $i_h = si_p$ for this equation because $N = sM$. An increase in n thus increases the accuracy of the simple transformation of the interest rates by using time units, see Figure 6. As in the previous case, applying the conformal discrete time interest rates defined in Eq. (7) does not improve the accuracy.

Example 4. Suppose a four month money flow $40(\$/4m)$ in continuous time with $r = 0.1(1/y)$. With simple transformation by the time units, the corresponding four and one month interest rates are $4/120(1/4m)$ and $1/120(1/m)$, respectively. Time is first measured in the units of $4m$ and then in the units of m. The present value of the money flow is in the first case

$$\int_{t_0}^{t_0+1} 40e^{-r\times(t-t_0)}dt = -\frac{40}{r}(e^{-r} - e^0) = 1200\left(1 - e^{-\frac{4}{120}}\right)$$
$$= 39.34(\$).$$

Notice that a marginal change in time, dt, is measured above in units $4m$. The complete form of the integrated factor is then $40(\$/4m) \times dt(4m)e^{-r\times(t-t_0)} = 40dte^{-r\times(t-t_0)}(\$)$ because $e^{-r\times(t-t_0)}$ is dimensionless.

Next time is measured in months. Flow $40(\$/4m)$ corresponds to $10(\$/m)$ and the present value is

$$\int_{t_0}^{t_0+4} 10e^{-r\times(t-t_0)}dt = -\frac{10}{r}(e^{-4r} - e^0) = 1200\left(1 - e^{-\frac{4}{120}}\right)$$
$$= 39.34(\$),$$

where dt is now measured in units m. Independent of the time unit applied, we get the same present value for the same flow in continuous time discounting. Essential is that the limits of integration, the interest rate, and the integrated flow are measured in equal units of time.

Example 5. We analyze the present value of the money flow in Example 4 in discrete time. Thus either $40(\$)$ is received after four months (the flow is $40(\$/4m)$ and the money is received at the ending moment of the time unit), or $10(\$)$ is received after every month in a four months time. The interest rate is $0.1(1/y)$ and we apply the simple transformation by using time units. The present value of the flow with $i_{4m} = 4/120$ is

$$\frac{40(\$/\Delta t) \times \Delta t}{1 + i_{4m} \times \Delta t} = \frac{40(\$/4m) \times 4m}{1 + \frac{4}{120}\left(\frac{1}{4m}\right) \times 4m} = 38.71(\$).$$

Next we calculate the present value of the monthly flow:

$$\sum_{j=1}^{4} \frac{10(\$/\Delta t)\Delta t}{(1 + i_m\Delta t)^j} = \frac{10(\$/m) \times (1/m)}{1 + \frac{1}{120}\left(\frac{1}{m}\right) \times (1m)} + \frac{10(\$/m) \times (1/m)}{\left(1 + \frac{1}{120}\left(\frac{1}{m}\right) \times (1m)\right)^2}$$
$$+ \frac{10(\$/m) \times (1/m)}{\left(1 + \frac{1}{120}\left(\frac{1}{m}\right) \times (1m)\right)^3} + \frac{10(\$/m) \times (1/m)}{\left(1 + \frac{1}{120}\left(\frac{1}{m}\right) \times (1m)\right)^4}$$
$$= 39.18(\$).$$

The more frequently the interest earnings are added in the capital in discrete time, the greater present value we thus get for a fixed positive flow. Thus the more dense the splitting of time in the discrete time, the more accurate approximate of it the continuous time calculated value is.

7. Analysis of Debt and Deposit Capitals

In order to simplify the solutions of the difference equations to be defined, we assume $t_0 = 0$ and that in discrete time either $\Delta t = 1(y)$ or $\Delta t = 1(m)$. Time intervals $\dots, t_0 + (n-1)\Delta t, t_0 + n\Delta t, t_0 + (n+1)\Delta t, \dots$ then become $\dots, n-1, n, n+1, \dots$, which simplifies the notation. We denote by $V(t_n)$ and V_n the debt and by $S(t_n)$ and S_n the deposit capital of a household in continuous and in discrete time at time moment t_n; these all have unit \$. The average and instantaneous velocities of the corresponding capitals are

$$\frac{\Delta V_n}{\Delta t},\ V'(t_n),\ \frac{\Delta S_n}{\Delta t},\ S'(t_n);$$

all these have unit $\$/y$ or $\$/m$. The interest costs (returns) of the debt (deposit) capitals during time unit y or m with unit $\$/y$ or $\$/m$, are then

$$iV_n,\ \ rV(t_n),\ \ iS_n,\ \ rS(t_n).$$

Let us now assume that the household repays his loan (or accumulates his savings) by a fixed amount of money B_y $(\$/y)$ or $B_m(\$/m)$ at every time unit Δt in all cases. For simplicity, the continuous and discrete time interest rates are assumed fixed. Then we get four equations describing the velocities of the capitals:

$$V'(t_n) = rV(t_n) - B_b \ \ or \ \ \frac{\Delta V_n}{\Delta t} = iV_n - B_b,$$

$$S'(t_n) = rS(t_n) + B_b \ \ or \ \ \frac{\Delta S_n}{\Delta t} = iS_n + B_b, \quad b = y, m.$$

where both sides of all equations either have unit $\$/y$ or $\$/m$. Because the equations are analogous, for shortness we study here only the debt equations. The discrete time debt equation is modified by writing $\Delta V_n = V_{n+1} - V_n$. Multiplying the equation by Δt, yields

$$V_{n+1} = (1 + i\Delta t)V_n - B_b\Delta t. \tag{10}$$

Notice that the definition of $\Delta V_n = V_n - V_{n-1}$ would give

$$(1 - i\Delta t)\overline{V}_n = \overline{V}_{n-1} - B_b\Delta t \iff \overline{V}_n = \overline{V}_{n-1} + i\Delta t\overline{V}_n - B_b\Delta t, \qquad (11)$$

where the bar above V deviates the two equations. Now, Eq. (10) is the correct form for the difference equation, because there on the left hand side is the debt capital at time moment $n + 1$, and the right hand side consists of the capital at time moment n added by the interest costs $i\Delta t V_n$ and subtracted by the repayment $B_b\Delta t$ at time n. Both sides thus equal with the capital at time moment $n + 1$. In Eq. (11), the interest costs are calculated by using the future, and not the current, value of the debt capital, which overestimates the speed of repayment.

Because $\Delta t, i$, and B_b are assumed fixed, Eqs. (10) and (11) are linear first order difference equations with constant coefficients. The solutions of the differential and difference equations are

$$V(t_n) = \left(V(0) - \frac{B_b}{r}\right)e^{rt_n} + \frac{B_b}{r}, \qquad (12)$$

$$V_n = \left(V_0 - \frac{B_b}{i}\right)(1 + i\Delta t)^n + \frac{B_b}{i}, \qquad (13)$$

$$\overline{V}_{\overline{n}} = \left(\overline{V}_0 - \frac{B_b}{i}\right)\left(\frac{1}{1 - i\Delta t}\right)^{\overline{n}} + \frac{B_b}{i}, \quad b = y, m, \qquad (14)$$

where $V(0), V_0$ and $\overline{V}_0$ are the initial conditions and the order of the time interval in the last equation is denoted by $\overline{n}$. The reader can check that these solutions are dimensionally well defined. If the debt capital is required to be paid off at a future time moment N, the initial conditions must be solved setting $V(N) = V_n = \overline{V}_{\overline{n}} = 0$.

Setting $V(t_n) = V_n$, we get a condition that gives identical results for the time path of the debt capital in continuous and in discrete time,

$$\left(V(0) - \frac{B_b}{r}\right)e^{rt_n} + \frac{B_b}{r} = \left(V_0 - \frac{B_b}{i}\right)(1 + i\Delta t)^n + \frac{B_b}{i}, \quad b = y, m. \quad (15)$$

An analytic solution for r in terms of i does not exist for Eq. (15). However, setting the numerical values of t_n and n equal gives numerical solutions for Eq. (15). These solutions reveal an essential dependence on n. The condition for equal time paths for the debt capital in discrete time with units $1(y) = 12(m)$ is,

$$\left(V_0 - \frac{B_y}{i_y}\right)\left(1 + i_y(1y)\right)^n + \frac{B_y}{i_y}$$
$$= \left(V_0 - \frac{B_m}{i_m}\right)(1 + i_m(1m))^{12n} + \frac{B_m}{i_m}. \tag{16}$$

Again, no analytic solution for i_y in terms of i_m (or vice versa) exists for Eq. (16). However, numerical solutions can be found that essentially depend on n. We can thus conclude that for the dynamics of debt and deposit capitals, exact transformation rules between the continuous and discrete time interest rate, and between discrete time interest rates with different time units, cannot be derived. The corresponding conformal interest rates must be solved from Eqs. (15) and (16) with known values of n.

Finally, we analyze the accuracy of Eq. (12) concerning the real world discrete time calculation in Eq. (13). Figures 8, 9 display the time paths of the debt capitals (dots refer to Eq. (13)) during 20 years (240 months) with annual and monthly data: $V(0) = V_0 = 5000(\$)$, $r = i = 0.1(1/y) = 1/120\,(1/m)$ and $B_y = 600(\$/y) = 600/12\,(\$/m) = B_m$. On annual basis, the difference of roughly one year exists between the two methods concerning the time moment the debt capital is zero. On monthly basis, the difference is less than a month. With discrete time by using $i_m = 1/120\,(1/m)$ and $i_y = 0.1(1/y)$, the time paths of the debt capitals are displayed in Figure 10 (dots refer to annual analysis). The simple rules $i = r$ and $i_y = 12i_m$ turn out to produce more accurate results for the debt capitals than using the transformation Eqs. (5) and (7).

Example 6. We study numerically the accuracy of Eq. (12) concerning Eq. (13) with $V(0) = V_0 = 5000(\$)$, $r = i = 1/120\,(1/m)$, $B_m = 400(\$/m)$, $t_n = 7(m)$ and $n = 7$. Then $V(7m) = 2417.1(\$)$ and $V_7 = 2428.1(\$)$. Setting $V(t_n) = V_n = 0$ and solving the time moment (the order of the time interval) this occurs, we get $t_n = 13.20(m)$ and $n = 13.26$. Eq. (12) thus produces results accurate enough for the monthly discrete time analysis.

8. Summary

We studied discrete and continuous time interest calculation with the aim to find an exact way to treat time in the analysis. We also searched for transformation rules between various interest rate concepts that would give identical results by using different calculation methods. Measurable concepts corresponding to instantaneous velocities of debt and deposit capitals were defined and applied in modeling time paths of the capitals. Clear transformation rules for compound interest calculation were found, which though is not a new result. However, we showed that for discounting money flows and calculating the time paths of debt and deposit capitals, such rules can not be found. Transforming interest rates by using time units gives more accurate results in these cases than applying the conformal interest rates defined for compound interest calculation.

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Appendix

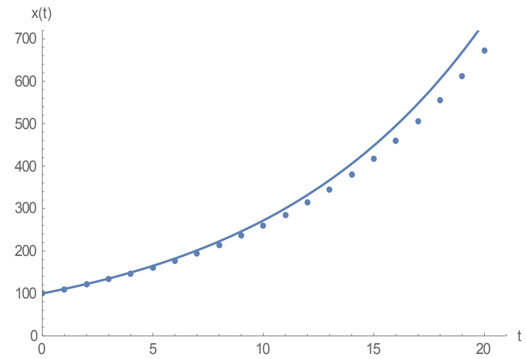

Figure 1. Capital $x(0) = 100(\$)$ in discrete and continuous time: $r = i = 0.1(1/y),\ \Delta t = 1(y)$.

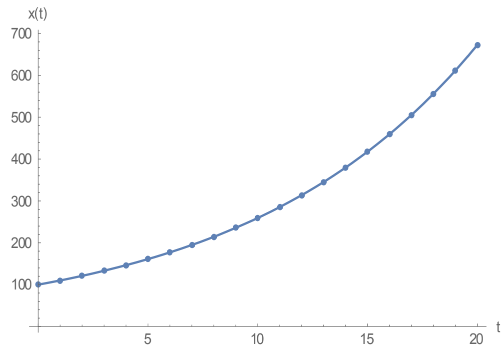

Figure 2. Capital $x(0) = 100(\$)$ in discrete and continuous time: $i \times (1l1y) = 0.1, r\Delta t = \ln(1 + i\Delta t)$.

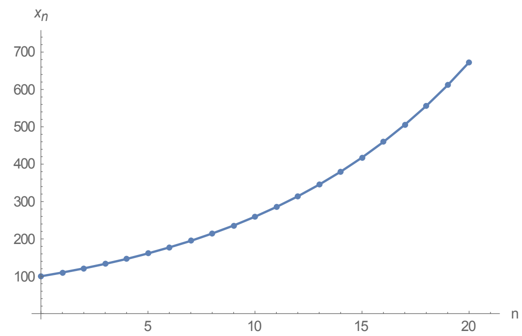

Figure 3. Capital $x(0) = 100(\$)$ with discrete time interest rates: $i_y \times (1/y) = 0.1,\ i_m \times (1/m) = 1/120$.

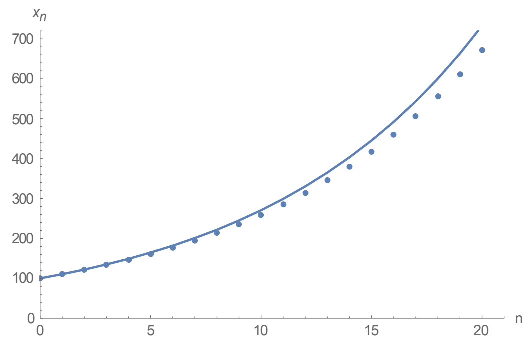

Figure 4. Capital $x(0) = 100(\$)$ in discrete time: $i_y \times (1/y) = 0.1,\ i_m \times (1/m) = \left(1 + i_y(1y)\right)^{1/12} - 1.$

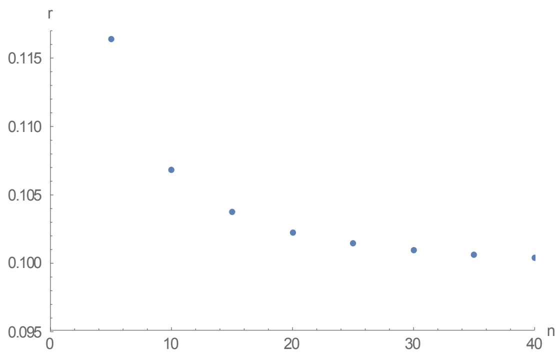

Figure 5. The continuous time conformal interest rates for $i_y \times (1/y) = 0.1$ with $n = 5, \ldots, 40$.

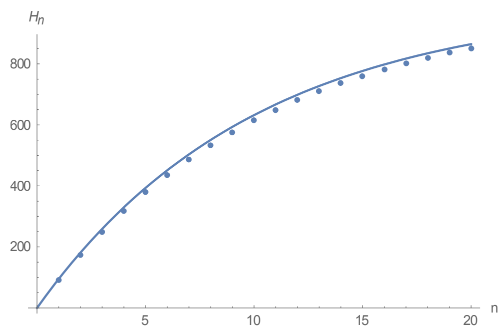

Figure 6. The present values of flow $100(\$/y)$ in discret and continuous time: $r = i = 0.1(1/y) = 0.1$, $\Delta t = 1(y)$.

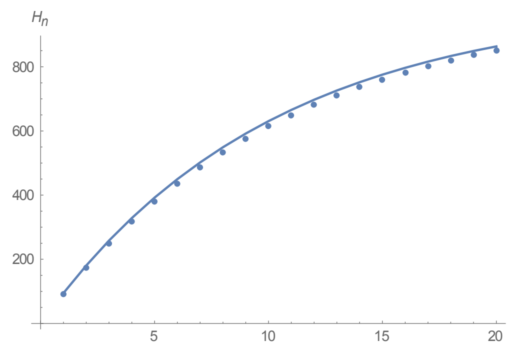

Figure 7. The present values for flow $100(\$/y)$ in discrete time: $i_y \times (1/y) = 0.1, i_m \times (1/m) = 1/120$.

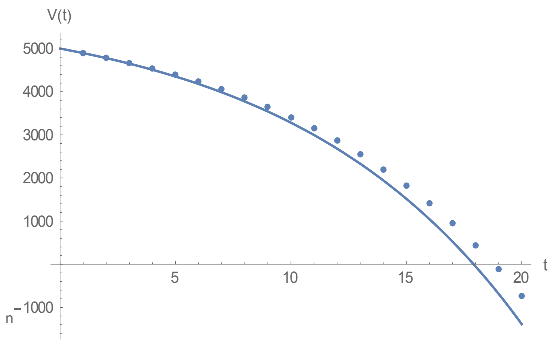

Figure 8. The debt capital in discrete and in continuous time: $r = i = 0.1(1/y), V(t_0) = V_0 = 5000(\$)$, $B = 600(\$/y)$.

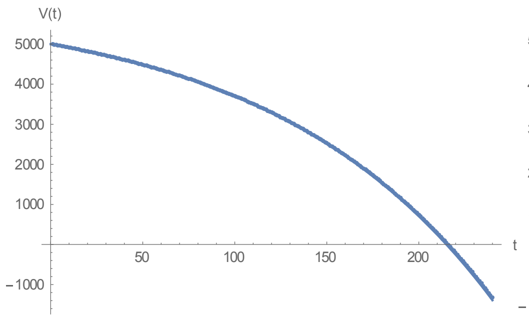

Figure 9. The debt capital in discrete and in continuous time $r = i = 1/120\,(1/m), V(t_0) = V_0 = 5000(\$)$, $B = 50(\$/m)$.

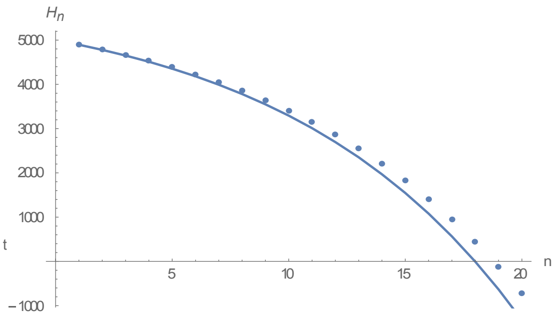

Figure 10. The debt capital in discrete time: $i_y \times (1y) = 0.1, i_m \times (1m) = 1/120$.