GEOMETRIC TRANSFORMATIONS. A LARGE STUDY OF THE ROTATIONS IN THE 3D SPACE. MATHEMATICAL MODELS. INFORMATICS MODELS.

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Abstract. *The work has several sections. The main section deals with the rotations in the space around each axis, with a given angle and around a given space vector. Each section contains the mathematical models and the informatics models.*

Introduction

This presentation is a summary of chapter 3 from the edited book: Nicolae Popoviciu. *Capitole fundamentale de matematică. Modele matematice ilustrate prin teorie, programe sursă si aplicatii.* (*Fundamental chapters of mathematics illustrated by theory, source program and applications*), Victor Publishing House, ISBN 978-606-754-011-6, Bucharest, 2017.

Chapter 3 contains 92 pages; Page 99-190. Geometric transformations. The rotations in plane. The rotations in space. Mathematical models. Informatics models. (92 pages; pag. 99-190). Chapter 3 contains 53 solved problems.

1. The rotation in the space, around of coordinates axis

1.1. Notations

$\{O, \vec{i}, \vec{i}, \vec{k}\}$ is a three-orthogonal system in R^3.

$P(x, y, z)$ is the initial general point, $P(X, Y, Z)$ is the rotated point.

$P(a, b, c)$ is the initial fixed point, $P(\alpha, \beta, \eta)$ is the rotated point.

The space R^3 has 8 octants I, II, III, IV, V, VI, VII, VIII.

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$$R_x = R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$ is the rotation matrix around Ox.

$$R_y = R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta \\ 0 & 0 & 1 \end{pmatrix}$$ is the rotation matrix around Oy.

$$R_z = R_x(\omega) = \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ is the rotation matrix around Oz.

Properties. $\det R_x = 1,\ R_x R_y^T = I,\ Trace R_x = 1 + 2\cos\phi.$

$\det(R_x - \lambda) = (1-\lambda)(\lambda^2 - 2\cos\phi\lambda + 1)$.

Eigen values $\lambda_0 = 1,\ \lambda_1 = \cos\phi + i\sin\phi,\ \lambda_2 = \cos\phi - i\sin\phi.$

The subspace $S_{R_x}(\lambda_0 = 1) = \left\{ u \in R^3 / u = \alpha_1 X_1\ X_1 = (100)^T \right\},\ R_x u = u.$

X_1 is the principal eigen vector.

1.2. The types of rotations in space. The symmetries in space

Applications.

P1. $P(1,1,0)$; the symmetry Ox yields $P'(1,-1,0)$.

P2. $P(1,1,1)$; the symmetry Ox yields $P'(1,-1,-1)$.

Generalization $P(a_1 > 0, a_2 > 0, a_3 > 0)$ yields $P'(a_1, -a_2, -a_3)$; repetition P'' etc.

P3. $P(1,1,1)$; the symmetry xOy yields $P'(1,1,-1)$.

Generalization $P(a_1 > 0, a_2 > 0, a_3 > 0)$ yields $P'(a_1, a_2, -a_3)$; repetition P'' etc.

P4. $P(1,1,1)$; the symmetry $O(0,0,0)$ yields $P'(-1,-1,-1)$.

1.3. The rotation with a given angle in relation with the axis Ox or Oy or Oz

1.3.1. The rotation with a given angle in relation with the axis Ox. The rotation matrix. Properties

The geometric rule of angle ϕ, related with Ox.

Direct problem. The point $P(a,b,c)$ and angle ϕ are given. Rotate this point around Ox and obtain $P'(\alpha,\beta,\eta)$ (with $\phi>0$ or $\phi<0$).)

Step1. Project $P(a,b,c)$ on Ox in point O', where $O'=O$ or $O'\neq O$.

Step 2. Construct the circle $\Gamma_x=\Gamma_x(O',r=O'P)$. The plane $\Gamma_x\perp Ox$.

Step 3. We measure the angle ϕ on Γ_x, from the point P.

Step 4. Calculate $R_x(\phi)\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}\alpha\\\beta\\\eta\end{pmatrix}$.

1.3.2. Applications for the axis Ox

P5. $P(0,1,0)\in Oy$. For $\phi=45^0$ find P' and again for $\phi=45^0$ find P''.

$P'\left(0,\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$, $P''\left(0,0,1\right)$).

P6. $P(1,1,0)\in xOy$. For $\phi=90^0$ find P' and again for $\phi=180^0$ find P''..

$P'\left(1,0,1\right)$, $P''\left(1,0,-1\right)$).

1.3.3. The rotation of the current point of the vector $\overline{OA}$ in relation with Ox axis. Generalization

Step 1. OA has the equations $\frac{x}{a_1}=\frac{y}{a_2}=\frac{z}{a_3}$, $A(a_1,a_2,a_3)\in R^3$, or parametric equations $x=a_1t,\ y=a_2t,\ z=a_3t,\ 0\le t\le 1$.

Step 2. The given current point $P(a,b,c)$ has $a = a_1 t,\ b = a_2 t,\ c = a_3 t,$ $0 \le t \le 1$.

Step 3. Calculate $R_x(\phi)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a_1 t \\ a_2 t\cos\phi - a_3 t\sin\phi \\ a_2 t\sin\phi + a_3 t\cos\phi \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \eta \end{pmatrix}.$

Application. $A(2,2,2),\ P(1,1,1),\ \phi = 45^0;\ t = \frac{1}{2}$ and $P'(1,0,\sqrt{2})$.

1.3.4. The rotation of the current point of the vector $\overline{AB}$ in relation with Ox axis. Generalization.

Given points. $A(a_1,a_2,a_3) \in R^3,$ $B(b_1,b_2,b_3) \in R^3,$ $P(a,b,c);$ $P'(\alpha,\beta,\eta)$ =?

Step 1. AB has the equations $\frac{x-a_1}{b_1-a_1} = \frac{y-a_2}{b_2-a_2} = \frac{z-a_3}{b_3-a_3}.$

Step 2. Condition $P \in (AB),$ $\frac{a-a_1}{b_1-a_1} = \frac{b-a_2}{b_2-a_2} = \frac{c-a_3}{b_3-a_3}.$

Step 3. Parametric coordinates of P are

$a = a_1 + t(b_1-a_1),\ b = a_2 + t(b_2-a_2),\ c = a_3 + t(b_3-a_3),\ 0 \le t \le 1.$

Step 4. Translate $\overline{AB}$ paralleled so that $A = O.$

$A \to A_1(0,0,0) = O(0,0,0,)$

$A \to A_1(a_1-a_1, a_2-a_2, a_3-a_3) = O(0,0,0,)$

$B \to B_1(b_1-a_1, b_2-a_2, b_3-a_3),\ P \to P_1(a-a_1, b-a_2, c-a_3).$

Step 5. Calculate $R_x(\phi)\begin{pmatrix} a-a_1 \\ b-a_2 \\ c-a_3 \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$ and rotate P_1 in $P_1^{'}(p,q,r),$ where

$p = a - a_1$

$q = (b-a_2)\cos\phi - (c-a_3)\sin\phi,\ r = (b-a_2)\sin\phi + (c-a_3)\cos\phi.$

Step 6. Calculate the inverse traslation of $P_1^{'}(p,q,r)$ and obtain $P'(\alpha,\beta,\eta)$ with

$\alpha = p + a_1 = a$

$\beta = q + a_2 = (b - a_2)\cos\phi - (c - a_3)\sin\phi + a_2$

$\eta = r + a_3 = (b - a_2)\sin\phi + (c - a_3)\cos\phi + a_3.$

The informatics model is based on the steps 1-6.
Numerical application.

P7. $A(a_1 = 0{,}5 \quad a_2 = 0{,}5 \quad a_3 = 0{,}5),\ B(b_1 = 2{,}5 \quad b_2 = 2{,}5 \quad b_3 = 2{,}5)$

$P(a = 1{,}5 \quad b = 1{,}5 \quad c = 1{,}5),\ \phi = 45^0$

$P'(\alpha, \beta, \eta) = P'(1{,}5 \quad 0{,}5 \quad \sqrt{2} + 0{,}5).$

P8. $A(a_1 = 1 \quad a_2 = 2 \quad a_3 = 3),\ B(b_1 = 5 \quad b_2 = 6 \quad b_3 = 7)$

$P(a = 3 \quad b = 4 \quad c = 5),\ \phi = -45^0$

$P'(\alpha, \beta, \eta) = P'(3 \quad 2\sqrt{2} + 2 \quad 3).$

1.3.5. The rotation with a given angle in relation with the axis *Ox,Oy,Oz.* The rotation matrix. Properties. Applications.

P9. On the axis Ox is given the fixed point $P(1,0,0).$

a) Rotate $P(1,0,0)$ with $\theta = 45^0$, and obtain P'.

b) Rotate P' with $\theta = 45^0$, and obtain P''.

c) Rotate P'' with $\theta = 90^0$, and obtain P'''.

Find the coordinates of the points P', P'' și P'''.

Solution. a) With geometrical illustration we project the point $P(1,0,0)$ on Oy in the point $O' = O$, $O'(0,0,0)$. Then we construct the circle $\Gamma_y = \Gamma_y(O'P)$.

We use the matrix $R_y(\theta = 45^0)$ and obtain

$$\begin{pmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 \\ 0 \\ -\sqrt{2}/2 \end{pmatrix}, P'\left(\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right).$$ The position is correctly in the plane xOz, with the negative axis Oz;

b) We use the matrix $R_y(\theta = 45^0)$ and obtain

$$\begin{pmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} \sqrt{2}/2 \\ 0 \\ -\sqrt{2}/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix},$$ $P''(0,0,-1)$. The position is correctly on the negative axis *Oz*.

c) With the matrix $R_y(\theta = 90^0)$ we obtain

$$\begin{pmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -\sqrt{2}/2 \\ 0 \\ -\sqrt{2}/2 \end{pmatrix},$$ $P''\left(-\sqrt{2}/2,\ 0, -\sqrt{2}/2\right)$. The result is correctly in the plane xOz.

P10. On the axis Oz is given the point $P(0,0,1)$.

a) Rotate $P(0,0,1)$ with angle $\theta = 45^0$, and obtain the position P'.

b) Rotate P' with the angle $\theta = 45^0$, and obtain P''.

Find the coordinates of the points P' and P''.

Solution. a) We project the point $P(0,0,1)$ on the axis Oy in the point $O' = O$, $O'(0,0,0)$. We construct the circle $\Gamma_y = \Gamma_y(O'P)$.

By using the matrix $R_y(\theta = 45^0)$ we obtain

$P'\left(\frac{\sqrt{2}}{2},\ 0, \frac{\sqrt{2}}{2}\right)$, in the plane xOz.

b) By using the matrix $R_y(\theta = 45^0)$ we obtain the point

$P''(1,\ 0,0)$. The result is correctly, in the positive axis Ox.

In the same manner we can study

The rotation of the current point of the vector $\overline{OP}$ in relation with Oy axis.

The rotation of the current point of the vector $\overline{AB}$ in relation with Oy axis.

The rotation with a given angle in relation with the axis Oz. The rotation matrix. Properties. Applications.

The point $P(a,b,c)$ don't belong to the axis Oz.
P11. On the axis Ox is given the point $P(1,0,0)$.

a) Rotate $P(1,0,0)$ with $\omega = 45^0$, and obtain the position P'.

b) Rotate P' with $\omega = 45^0$, and obtain the position P''.

c) Rotate P'' with $\omega = 90^0$, and obtain the position P'''.
Find the coordinates of the points P', P'' and P'''.

Solution. a) We project the point $P(1,0,0)$ on the axis Oz in the point $O' = O$, $O'(0,0,0)$. We construct the circle $\Gamma_z = \Gamma_z(O'P)$.
By using the matrix $R_z(\omega = 45^0)$ we obtain

$$\begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix}, P'\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right).$$ The position of the result is correctly in the plane xOy, with the positive axis Oy.

b) We use the matrix $R_z(\omega = 45^0)$ and obtain

$$\begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, P''(0,1,0)$$ in correctly on Oy.

c) By using the matrix $R_z(\omega = 90^0)$ we obtain

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, P'''(-1,0,0).$$ Correctly on negative axis Ox.

P12. On tha axis Oy we have the fixed point $P(0,1,0)$.
a) Rotate $P(0,1,0)$ with angle $\omega = 45^0$ and obtain the position P'.

b) Rotate P' with angle $\omega = 45^0$ and obtain P''.

Find the coordinates of the points P' and P''.

Solution. a) We project $P(0,1,0)$ on Oz, in the point $O' = O$, $O'(0,0,0)$ and then construct the circle $\Gamma_z = \Gamma_z(O'P)$.

Use the matrix $R_z(\omega = 45^0)$ and obtain $P'\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$. The position is correctly in the plane xOy.

b) Use the matrix $R_z(\omega = 45^0)$ and obtain $P''(-1, 0, 0)$, correctly on the negative axix Ox. .

P13. In the space $\mathbf{R}^3$ we have the point $P(1,1,1)$.

a) Rotate $P(1,1,1)$ with angle $\omega = 45^0$ and obtain P'.

b) Rotate the same point $P(1,1,1)$ with $\omega = -45^0$ and obtain P''.

c) Rotate the point P' c with $\omega = 90^0$ and obtain the position P'''.

Find the coordinates of the points P', P'' and P'''.

Solution. a) We project the point $P(1,1,1)$ on the axis Oz in the point $O' \neq O$, $O'(0,0,1)$. Construct the circle $\Gamma_z = \Gamma_z(O'P)$.

Use the matrix $R_z(\omega = 45^0)$ and obtain the point $P'(0, \sqrt{2}, 1)$ in the plane yOz.

b) Use the matrix $R_z(\omega = -45^0)$ and obtain $P''(\sqrt{2}, 0, 1)$ in the plane xOz.

c) Use the matrix $R_z(\omega = 90^0)$ and obtain $P'''(0, \sqrt{2}, 1)$ in the plane yOz..

In the same manner we can study

The rotation of the current point of the vector $\overline{OP}$ related with Oz axis.

The rotation of the current point of the vector $\overline{AB}$ related with Oy axis.

The rotation in the space, around of one given vector.

The combined rotation containing three angles related with one axis.

2. The inverse problem. The obtaining of rotation axis and rotation angle from matrix rotation. The inverse rotation algorithm. Applications.

2.1. The inverse problem means the obtaining of rotation axis and rotation angle from matrix rotation

2.2. The inverse rotation algorithm. Applications

Direct Problem. From rotation axis u and the rotation angle ϕ we obtain the rotation matrix $R = R_u(\phi)$, where $\|u\| = 1$ is the versor of u.

Inverse Problem. The rotation matrix $R = R_u(\phi) = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is known.

$\det R = 1, RR^T = I, R^{-1} = R, TraceR = 1 + \cos\phi, P_u(\lambda) = \det(R - \lambda I).$ Find u and ϕ.

The algorithm for finding u

Method 1. Step 1. Calculate the eigen values of $R = R_{3x3}$.

Step 2. Find the subspace $S_R(\lambda_0 = 1) = \{X / X = \alpha X_1\},\ \alpha \in \mathbf{R}$.

Step 3. Calculate the versor $u = \dfrac{X_1}{\|X_1\|}$.

Method 2. Step 1. Calculate the vector $v = \begin{pmatrix} h - f \\ c - g \\ d - b \end{pmatrix}$.

Step 2. Calculate $u = \dfrac{v}{\|v\|}$. (for informatics model).

The informatics model is based on the given matrix *R*, the vector *v* and the next algorithm.

The algorithm for finding ϕ.

Method 1. Step 1. Solve the trigonometric equation $TraceR = 1 + 2\cos\phi$.

Generally we obtain ϕ_1 and ϕ_2.

Step 2. Calculate the matrix $R_1 = R_u(\phi_1)$ and $R_2 = R_u(\phi_2)$ and compare with initial matrix.

Method 2. Step 1. Solve the trigonometric equation $\|v\| = 2\sin\phi$.

Step 2. Calculate the matrix $R_1 = R_u(\phi_1)$ and $R_2 = R_u(\phi_2)$ and compare with initial matrix.

P14. Application. The matrix R is $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -1/2 & \sqrt{3}/2 \end{pmatrix}$.

a) Verify the properties of R.
b) Find the eigen values.
c) Find the real subspaces.
d) Find the rotation axis u.
e) Find the rotation angle ϕ.
f) Verify the correctitude of computations.

Solution. $P_R(\lambda) = (1-\lambda)(\lambda^2 - \lambda\sqrt{3} + 1), \quad P_R(\lambda) = 0$

$\lambda_0 = 1, \ \lambda_1 = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \ \lambda_2 = \frac{\sqrt{3}}{2} - \frac{1}{2}i.$

$RX = \lambda_0 X, \ X = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}, \ X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$

Find $u = \frac{X_1}{\|X_1\|} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Ox axis.

Find ϕ. Method 1. $Trace R = 1 + 2\cos\phi, \ \phi_1 = 30^0 = \frac{\pi}{6}, \ \phi_2 = 330^0 = \frac{11\pi}{6}.$

Find $R_x(\phi)$.

$R_1 = R_x(\phi_1 = 30^0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{pmatrix}$ (false).

$$R_2 = R_x(\phi_2 = 330^0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -1/2 & \sqrt{3}/2 \end{pmatrix} \text{ (true).}$$

Conclusions. The rotations study of any point or vector contains a lot of aspects. The two big directios are the rotations in the plane 2D and then the rotations in the space 3D. This work is a large study in 3D. A colmplete study in the case of 3D is contained in the book mentioned in Introduction.

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